COPY NO.

IMM-NYU 238
JANUARY 1957

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MATHEMATICAL SCIENCES

The Spinor Transformations of Maxwell's Equations

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PREPARED UNDER
CONTRACT DA-30-069-ORD-1258
WITH THE
OFFICE OF ORDNANCE RESEARCH
UNITED STATES ARMY

IMM-NYU 238
January 1957

THE SPINOR TRANSFORMATIONS OF MAXWELL'S EQUATIONS

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This report represents results obtained at the Institute of Mathematical Sciences, New York University, under the sponsorship of the Office of Ordnance Research, United States Army, Contract DA-30-069-ORD-1258.

New York, 1957

Abstract

A relativistic spinor formulation of Maxwell's equations has been set up and its transformation properties given explicitly.

Table of Contents

1. Introduction and Summary.

Previous authors, e.g. [1] [2], who have given Maxwell's equations in terms of a spinor notation have not concerned themselves with the relativistic invariance of these equations. In the present paper we extend the work of the previous authors, in particular the work of Molière, in such a way that the equations are relativistically invariant in spinor notation. It will turn out that one obtains the same transformation rules for field strengths as one obtains from the more conventional form of relativistic electrodynamics. However, the transformations will be somewhat simpler in form and will constitute an orthogonal representation of the proper Lorentz group. The transformations are summarized in section 4. The remainder of the paper deals with the proof of the group properties.

When there are no external sources, it will be seen that Maxwell's equations reduce to a special case of Dirac's equations for a particle of zero mass. The transformations of the spinor representing the Maxwell field differ, however, from that of the Dirac field. The transformation of the Maxwell field is such that the field has a spin 1, as is to be expected, whereas the Dirac field has spin 1/2.

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[1] G. Molière, Annalen Der Physik, 6. Folge, Band 6, P.146 (1949).

[2] A.D. Bressler and N. Marcuvitz, Operator Methods in Electro-Magnetic Field Theory, Research Report R-495-56, PIB-425. Microwave Research Institute of Polytechnic Institute of Brooklyn, (1956).

2. The Molière Formalism.

Maxwell's equations are

$$\text{(2.1)} \qquad \operatorname{div} \underset{\sim}{E} = 4\pi\rho \quad , \quad \operatorname{div} \underset{\sim}{H} = 0 \quad ,$$

$$\text{(2.2)} \qquad \frac{\partial \underset{\sim}{E}}{\partial t} - \operatorname{curl} \underset{\sim}{H} = -4\pi j \quad , \quad \frac{\partial \underset{\sim}{H}}{\partial t} + \operatorname{curl} \underset{\sim}{E} = 0$$

where ρ and j are charge and current densities.

Moliere introduces the three component column vectors which in our notation are

$$\text{(2.3)} \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \quad , \qquad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix}$$

where

$$\text{(2.3a)} \qquad \begin{aligned} \psi_1 &= H_x - iE_x & \qquad \Phi_1 &= j_x \\ \psi_2 &= H_y - iE_y & \Phi_2 &= j_y \\ \psi_3 &= H_z - iE_z & \Phi_3 &= j_z \end{aligned} \quad .$$

Then Maxwell's equations (2.2) are given by

$$\text{(2.4)} \qquad \left[-\frac{1}{i}\frac{\partial}{\partial t} - \frac{1}{i}\zeta^1 \frac{\partial}{\partial x} - \frac{1}{i}\zeta^2 \frac{\partial}{\partial y} - \frac{1}{i}\zeta^3 \frac{\partial}{\partial z} \right]\psi = -4\pi\Phi$$

where ζ^1, ζ^2, ζ^3 are matrices acting on the vector and are given by

$$\text{(2.5)} \qquad \zeta^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \zeta^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$\zeta^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad .$$

Now, it is to be noted that only Maxwell's equations (2.2) have been given in the "spinor" form (2.5). In conventional relativistic treatments it is shown that in order to have relativistic invariance, equations (2.1) must be included. Hence, we modify Molière's treatment in the following section.

3. <u>The Spinor Form for all of Maxwell's Equations</u>.

We introduce <u>four</u>-component column vectors

$$(3.1)\qquad \psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \qquad \underline{\Phi} = \begin{pmatrix} \underline{\Phi}_0 \\ \underline{\Phi}_1 \\ \underline{\Phi}_2 \\ \underline{\Phi}_3 \end{pmatrix},$$

where, as before,

$$\begin{aligned} \psi_1 &= H_x - iE_x & \qquad \underline{\Phi}_1 &= j_x \\ \psi_2 &= H_y - iE_y & \underline{\Phi}_2 &= j_y \\ \psi_3 &= H_z - iE_z & \underline{\Phi}_3 &= j_z \end{aligned} \quad .$$

In addition, however, we have

$$(3.2a)\qquad \psi_0 = 0 \qquad \underline{\Phi}_0 = \rho \quad .$$

Then <u>both</u> sets of Maxwell's equations can be written

$$(3.3)\qquad \left[-\frac{1}{i}\frac{\partial}{\partial t} - \frac{1}{i}\alpha^1\frac{\partial}{\partial x} - \frac{1}{i}\alpha^2\frac{\partial}{\partial y} - \frac{1}{i}\alpha^3\frac{\partial}{\partial z}\right]\psi = -4\pi\underline{\Phi}$$

where α^1, α^2, α^3 are matrices which act on ψ and are given by

$$(3.4)\qquad \alpha^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\alpha^3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} .$$

It is seen that the matrices α^i are Hermitian and it is easily shown by matrix multiplication that

$$(3.5) \qquad \begin{cases} (\alpha^1)^2 = (\alpha^2)^2 = (\alpha^3)^2 = I , \\ \alpha^1\alpha^2 = i\alpha^3 = -\alpha^2\alpha^1 , \\ \alpha^2\alpha^3 = i\alpha^1 = -\alpha^3\alpha^2 , \\ \alpha^3\alpha^1 = i\alpha^2 = -\alpha^1\alpha^3 , \end{cases}$$

where I is the identity operator

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Hence we mayalso write

$$(3.5a) \qquad \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^i{}_j \, I \quad , \qquad (i, j = 1, 2, 3)$$

where $\delta^i{}_j$ is the usual Kronecker δ given by

$$\delta^i{}_j = 0 \quad , \qquad i \neq 0$$
$$\delta^i{}_i = 1$$

In the Dirac theory of the electron one introduces a four component vector χ which satisfies the equation

$$(3.6) \qquad \left[- \frac{1}{i}\frac{\partial}{\partial t} - \frac{1}{i}\,\alpha^1 \frac{\partial}{\partial x} - \frac{1}{i}\,\alpha^2 \frac{\partial}{\partial y} - \frac{1}{i}\,\alpha^3 \frac{\partial}{\partial z} - \beta m \right]\chi = 0$$

where m is the mass of the electron and α^i are Hermitian matrices required to satisfy (3.5a) and β is a Hermitian matrix which satisfies

$$(3.6a) \qquad (\beta)^2 = I \quad , \quad \beta\alpha^i + \alpha^i\beta = 0 \quad .$$

It can be shown that matrices α^1, α^2, α^3, β which satisfy (3.5a) and (3.6a) are unique within a similarity transformation. In particular, one may choose the matrices α^i to be given by (3.4).

When $\underline{\Phi} = 0$, Maxwell's equations (3.3) take the form of Dirac's equation (3.6) corresponding to zero mass, and hence a solution of Dirac's equation subject to initial conditions which insure $\psi_0 = 0$ can be interpreted as a solution of Maxwell's equations when there are no external sources.

4. Relativistic Notation.

It is now convenient to introduce relativistic notation. Let us define, as is customary, the contravariant components of the world four-vector by

$$(4.1) \qquad x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z \quad .$$

The corresponding covariant components of the vector are given by

$$(4.1a) \qquad x_0 = t, \quad x_1 = -x, \quad x_2 = -y, \quad x_3 = -z \quad .$$

The length of the four-vector is defined to be

$$(4.2) \qquad x^i x_i = t^2 - x^2 - y^2 - z^2 \quad .$$

where here and elsewhere repeated indices mean summation over $i = 0, 1, 2, 3$ unless the indices are enclosed in parenthesis.

A Lorentz transformation assigns to the contravariant components of the x^i new components denoted by $x^{i'}$ where

$$x^{i'} = a^i{}_j \, x^j \quad , \tag{4.3}$$

$a^i{}_j$ being the matrix components of the transformation. The co-variant components transform in accordance with the rule

$$x'_i = a_i{}^j \, x_j \tag{4.4}$$

where

$$a_i{}^j = g_{ik} \, g^{jm} \, a^k{}_m \tag{4.5}$$

In (4.5) g_{ij} and g^{ij} are matrices whose components are given by

$$g_{ij} = g^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad .$$

One requires the length of a world vector to be invariant under a Lorentz transformation. Hence

$$x^{i'} \, x'_i = x^i \, x_i \tag{4.6}$$

This requirement leads to "orthonormality condition" on the Lorentz transformation

$$a_i{}^k \, a^i{}_j = \delta^k{}_j \quad . \tag{4.7}$$

Generally, a vector V is defined so that it has a set of contravariant components V^i and covariant components V_i, these

sets of components being related by

$$V^i = g^{ij} V_j$$

$$V_i = g_{ij} V^j$$

Under a Lorentz transformation the contravariant components V^i transform to new components $V^{i'}$ in accordance with the rule (4.3), while the covariant components V_i transform to V_i' in accordance with (4.4). The inner product of two vectors W and V, denoted by (W,V), is defined to be

$$(W,V) = V^i W_i = V_i W^i \tag{4.9}$$

The orthonormality condition (4.7) assures us that in the new representation (W,V) has the same value as in the old, i.e.

$$(W,V) = V^i W_i = V^{i'} W_i' \tag{4.10}$$

It is particularly to be noted that the operator $\frac{\partial}{\partial x^i}$ transforms like the covariant components of a vector.

To write Maxwell's equations (3.3) in relativistic form, we define α^0 to be

$$\alpha^0 = I \quad . \tag{4.11}$$

Then (3.3) takes the form

$$-\frac{1}{i}\,\alpha^j \frac{\partial}{\partial x^j}\,\psi = -4\pi\Phi \quad . \tag{4.12}$$

Under the Lorentz transformation

$$x^{i'} = a^i{}_j\, x^j \tag{4.4}$$

equation (4.12) takes the form

$$(4.13) \qquad -\frac{1}{i}\, \alpha^{j'} \frac{\partial}{\partial x^{j'}} \psi = -4\pi\underline{\Phi} \quad ,$$

where

$$(4.14) \qquad \alpha^{i'} = a^{i}{}_{j}\, \alpha^{j} \quad ,$$

that is,

$$\alpha^{i'} \frac{\partial}{\partial x^{i'}} = \alpha^{i} \frac{\partial}{\partial x^{i}}$$

behaves like an inner product, the matrices α^i being contravariant components of a vector and the derivatives $\frac{\partial}{\partial x^i}$ being the covariant components of another vector.

The requirements of relativistic invariance is that Maxwell's equations retain the same form in the new coordinate system as in the old. Thus, in the new coordinate system we require

$$(4.15) \qquad -\frac{1}{i}\, \alpha^{j} \frac{\partial}{\partial x^{j'}} \psi' = -4\pi\underline{\Phi} \quad ,$$

$$(4.15a) \qquad \psi_0{}' = 0 \quad .$$

where $\psi_0{}'$ is the topmost component of the column vector ψ'.

As an Ansatz, let us take

$$(4.16) \qquad \psi' = S\, \psi$$

where S is a matrix

$$(4.16a) \qquad S = (S_{ij}) \quad , \qquad (i, j = 0, 1, 2, 3)$$

and, since there is no a priori reason to assume $\underline{\Phi}'$ is related to $\underline{\Phi}$ as ψ' is to ψ, let us write

(4.17) $$\underline{\Phi}' = T\,\underline{\Phi}$$

(4.17a) $$T = (T_{ij}) \quad , \qquad (i, j = 0, 1, 2, 3)$$

where T is another matrix. Equation (4.17) is equivalent to giving the transformation properties of $\alpha^i \frac{}{x^i} \psi$, namely

(4.17b) $$\alpha^{i'} \frac{\partial}{\partial x^{i'}} \psi' = T\, \alpha^i \frac{\partial}{\partial x^i} \psi \quad .$$

We may regard (4.17b) as defining T rather than (4.17) when $\underline{\Phi} = 0$. That is, (4.17b) defined T in a more general fashion than (4.17).

Now, the condition (4.15a) implies a condition on S, namely

(4.18) $$S_{01} = S_{02} = S_{03} = 0 \quad .$$

Furthermore (4.17) and (4.17a) implies T is a <u>real</u> matrix, since the components of $\underline{\Phi}'$ and $\underline{\Phi}$ are real.

On using (4.16), (4.17) or (4.17a), (4.15) and (4.13) we have

(4.19) $$T^{-1}\alpha^i\, S = \alpha^{i'} \quad , \qquad (i = 0, 1, 2, 3)$$

or on using $(\alpha^i)^2 = I$,

(4.20) $$S = \alpha^{(i)}\, T\alpha^{(i)'} = a^i{}_j\, \alpha^{(i)} T \alpha^j \quad , \quad (i = 0, 1, 2, 3)$$

where the parenthesis around the superscripts mean that the summations are <u>not</u> to be taken with respect to these superscripts.

It is important to note that we require the <u>same</u> matrices S and T to satisfy each of the four equations (4.20). That it is possible to find S and T which obey this requirement is the

principle result of this paper which is the following:

> Under the conditions of (4.18) on S and the reality condition on T the matrices S of (4.20) form an orthogonal representation of the proper Lorentz group; the matrices T form a real representation of the Lorentz group. In fact
>
> (4.21) $$T_{ij} = a^{i}{}_{j} \quad .$$
>
> Having (4.21) at our disposal we can solve for S using any of the four equations (4.20).

Incidentally, the relation (4.21) says that the vectors Φ transform like the contravariant components of a four-vector. This result is, of course, in agreement with the more conventional relativistic treatment of Maxwell's equations.

It is seen that the matrices S have to be calculated from the operators $\alpha^i\ T\ \alpha^j$. The matrix forms of these operators are given in Appendix I for convenience, since they will be often referred to.

5. Some Properties of the Transformation Operators S and T.

As mentioned above, the matrix S as calculated by any of the four equations (4.20) must be the same. When i = 0, a particularly simple equation results, namely

(5.1) $$S = a^{0}{}_{j}\ T\ \alpha^{j} \qquad ,$$

since $\alpha^0 = I$.

If the rotation is completely space-like, then

(5.2) $$a^{0}{}_{j} = \delta^{0}{}_{j}$$

and hence, from (4.21) and (5.1)

$$(5.3) \qquad S = T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^1{}_1 & a^1{}_2 & a^1{}_3 \\ 0 & a^2{}_1 & a^2{}_2 & a^2{}_3 \\ 0 & a^3{}_1 & a^3{}_2 & a^3{}_3 \end{pmatrix}$$

Thus ψ_1, ψ_2, ψ_3 transform as components of vectors in ordinary three-dimensional space, and consequently, the electric field $\underset{\sim}{E}$ and magnetic field $\underset{\sim}{H}$ also transform as three-dimensional vectors under a transformation of space coordinates.

More generally, we can write S explicitly in terms of $a^i{}_j$ using (5.1), (4.21) and the results of Appendix I. The elements S_{ij} so obtained are written out explicitly in Appendix II. It is seen that each element S_{ij} consists of a complex number, the real and imaginary parts of which are each 2-rowed minors of the matrix $T = (a^i{}_j)$.

If one used another equation of (4.20), say i = 1 instead of i = 0, to find S, some of the matrix elements of S would be expressed in terms of different minors than before. For example, if one uses i = 0 in (4.20), one has

$$S_{22} = a^2{}_2\, a^0{}_0 - a^2{}_0\, a^0{}_2 + i(a^2{}_2\, a^0{}_1 - a^2{}_1\, a^0{}_3) \quad ,$$

while on using i = 1, the same matrix element has the form

$$S_{22} = a^1{}_1\, a^3{}_3 - a^1{}_3\, a^3{}_1 + i(a^1{}_2\, a^3{}_0 - a^1{}_0\, a^3{}_2) \quad .$$

At first sight it might seem that we have a contradiction and

that the equations (4.20) for S are inconsistent. However, the two expressions are in fact equal. The equality arises from the fact that the Lorentz transformations can be written as products of the transformation in planes. As a consequence, many of the various minors of T can be shown to be equal. This point will be discussed more fully in Section 7 where it will be shown that the equations (4.20) have unique solutions for T and S, that for T being given by (4.21).

6. The Group Properties of S and T.

We shall prove in the next section, Section 7, that S and T are given by equations (4.20) and (4.21) which for convenience we repeat below

$$S = a^{(i)}{}_{j}\, \alpha^{(i)}\, T\, \alpha^{j} \quad , \qquad (i = 0,\ 1,\ 2,\ 3) \tag{6.1}$$

$$T_{ij} = a^{i}{}_{j} \quad . \tag{6.2}$$

In this section we wish to prove that if S and T are indeed given as above, each matrix constitutes a representation of the (proper) Lorentz group.

Toward this end we must show the following:

(I) If the Lorentz transformation is the identity, then S and T are identity operators.

(II) If $\overset{1}{S}$ and $\overset{2}{T}$ correspond to the Lorentz transformation $\overset{1}{L}$, and $\overset{2}{S}$ and $\overset{2}{T}$ correspond to the Lorentz transformation $\overset{2}{L}$, then

the operators S and T which correspond to the Lorentz transformation $L = \overset{2}{L}\,\overset{1}{L}$ are given by $S = \overset{2}{S}\,\overset{1}{S}$, $T = \overset{2}{T}\,\overset{1}{T}$.

(III) If S and T correspond to L, S^{-1} and T^{-1} must correspond to L^{-1}.

We shall show first that if T is given by (6.2), then it satisfies the three requirements above.

First, if L is the identity transformations, then

(6.3) $$a^i{}_j = \delta^i{}_j = T_{ij} \quad ,$$

or

(6.3a) $$T = I \quad ,$$

as required.

Secondly, if

$$L = \overset{2}{L}\,\overset{1}{L} \quad ,$$

then

(6.4) $$a^i{}_j = \overset{2}{a}{}^i{}_k \, \overset{1}{a}{}^k{}_j \quad ,$$

where the transformation matrices $a^i{}_j$, $\overset{2}{a}{}^i{}_j$, $\overset{1}{a}{}^i{}_j$ correspond to L, $\overset{2}{L}$, $\overset{1}{L}$ respectively. Hence, from (6.2) we have

(6.5) $$T_{ij} = \overset{2}{T}_{ik} \, \overset{1}{T}_{kj}$$

or

(6.5a) $$T = \overset{2}{T}\,\overset{1}{T}$$

as required, where T, $\overset{2}{T}$, $\overset{1}{T}$ correspond to L, $\overset{2}{L}$, $\overset{1}{L}$.

The requirement III is proved similarly.

We shall now prove that requirements (I-III) are satisfied by S when S is given by (6.1) for all i.

We note first that the case where the Lorentz transformation is space-like includes the case where the Lorentz transformation is the identity. Hence, on using (6.3) for $a^i{}_j$, in (5.3) for S we have immediately (6.6) S = I, when L is the identity.

To show requirement II is valid, we write

$$
\begin{aligned}
S &= a^{(i)}{}_j\, \alpha^{(i)}\, T\, \alpha^j \quad , \\
\overset{1}{S} &= \overset{1}{a}{}^{(i)}{}_j\, \alpha^{(i)}\, \overset{1}{T}\, \alpha^j \quad , \qquad (6.7) \\
\overset{2}{S} &= \overset{2}{a}{}^{(i)}{}_j\, \alpha^{(i)}\, \overset{2}{T}\, \alpha^j \quad , \qquad (i = 0,\ 1,\ 2,\ 3)
\end{aligned}
$$

On using (6.4) and (6.5a) in the first of equations (6.7), we have

$$
\begin{aligned}
(6.8) \qquad S &= \overset{2}{a}{}^{(i)}{}_k\, \overset{1}{a}{}^k{}_j\, \alpha^{(i)}\, \overset{2}{T}\, \overset{1}{T}\, \alpha^j \\
&= \overset{2}{a}{}^{(i)}{}_k\, \alpha^{(i)}\, \overset{2}{T}\, \overset{1}{a}{}^k{}_j\, \overset{1}{T}\, \alpha^j \\
&= \overset{2}{a}{}^{(i)}{}_k\, \alpha^{(i)}\, \overset{2}{T}\, \alpha^k\, \alpha^k\, \overset{1}{a}{}^k{}_j\, \overset{1}{T}\, \alpha^j \\
&= \overset{2}{a}{}^{(i)}{}_k\, \alpha^{(i)}\, \overset{2}{T}\, \alpha^k\, \overset{1}{a}{}^k{}_j\, \alpha^k\, \overset{1}{T}\, \alpha^j \\
&= \overset{2}{S}\, \overset{1}{S}
\end{aligned}
$$

as required. In (6.8) we have used $(\alpha^i)^2 = I$.

Actually we have proved something stronger, namely that the same matrix S is given by all four of the equations $i = 0,\ 1,\ 2,\ 3$ when $\overset{1}{S}$ and $\overset{2}{S}$ as obtained from their corresponding four equations are the same.

The proof that requirement III is satisfied is only a minor variation of the proof given above and we therefore forego it.

7. Evaluation of S and T.

In this section we shall prove the following: The equations

$$S = a^{(i)}{}_{j}\, a^{(i)}\, T\, a^{j} \quad , \quad (i = 0, 1, 2, 3) \tag{7.1}$$

have unique solutions for S and T under the conditions

(a) $S_{01} = S_{02} = S_{03} = 0$.

(b) The matrices T constitute a real, continuous, single valued representation of the proper Lorentz group.

(c) When the Lorentz transformation is a transformation which involves only the time and one space coordinate so that the space axis of the transformed coordinates can be considered as moving parallel to the space axis of the origin coordinates with a velocity v with respect to the original system, we shall require that the absolute values of the components of Φ' (the transformed sources) depend only on the absolute value of v.

Further, we shall obtain the facts that

$$T_{ij} = a^{i}{}_{j} \tag{7.2}$$

and that the matrix S obtained from (7.1) will be the same whatever equation is used and will be orthogonal.

These results together with the results of Section 6 will prove that S and T form a representation of the Lorentz group.

Perhaps, requirement (c) above ought to be discussed somewhat. This requirement seems to be justified physically on the grounds that the sign of the relative velocity v should not influence the absolute magnitude of physically observable quantities, since this sign depends merely on the choice of the positive direction of the space axis.

If we gave up requirements (b) and (c) we should find that S and T are unique within multiplicative factors which form a one-dimensional representation of the Lorentz group.

We shall now proceed to prove our assertions. As the basis of our proof we shall use the fact that every proper Lorentz transformation can be written as a product of transformations, each involving two axes only. The number of such transformations, called transformations in the plane, needed to constitute an arbitrary transformation is at most six of which, at most one is a transformation involving a space coordinate and the time coordinate [3].

Let us first consider Lorentz transformations consisting of only a single rotation in a plane and of these we shall consider space rotations, such as rotations involving the x^1, x^2 coordinates. We then have

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[3] H.C. Lee, Quarterly Journal of Mathematics, 15, 7, (1944).

$$(7.3)\qquad \begin{cases} x^{1'} = x^1 \cos\Theta - x^2 \sin\Theta \\ x^{2'} = x^1 \sin\Theta + x^2 \cos\Theta \\ x^{0'} = x^0 \\ x^{3'} = x^3 \quad . \end{cases}$$

Hence

$$(7.4)\qquad \begin{cases} a^1{}_1 = \cos\Theta = a^2{}_2 \qquad a^1{}_2 = -\sin\Theta = -a^2{}_1 \\ a^0{}_0 = 1 \qquad a^3{}_3 = 1 \\ \text{all other } a^i{}_j = 0 \ . \end{cases}$$

We write the four equations of (7.1) as

$$(7.5a)\qquad S = T \quad ,$$

$$(7.5b)\qquad S = \alpha^3\ T\ \alpha^3 \quad ,$$

$$(7.5c)\qquad S = \cos\Theta\ \alpha^1\ T\ \alpha^1 - \sin\Theta\ \alpha^1\ T\ \alpha^2 \quad ,$$

$$(7.5d)\qquad S = \sin\Theta\ \alpha^2\ T\ \alpha^1 + \cos\Theta\ \alpha^2\ T\ \alpha^2 \quad .$$

Let us define $b(\Theta)$ by

$$(7.6)\qquad S_{00} = b(\Theta)$$

where we indicate explicitly the dependence of S_{00} on Θ.

On using condition (a) of page 15, (namely $S_{01} = S_{02} = S_{03} = 0$), equation (7.6), and (7.5a) we see

$$(7.7)\qquad T_{00} = b(\Theta) \quad , \quad T_{01} = T_{02} = T_{03} = 0 \quad .$$

On using (7.5b) and Appendix I for the matrix elements of $\alpha^3\ T\ \alpha^3$ we have

$$(7.8)\qquad T_{33} = b(\Theta) \quad , \quad T_{30} = T_{31} = T_{32} = 0 \quad .$$

On using (7.5c) and Appendix I we obtain

$$(7.9)\qquad \begin{cases} b(\theta) = \cos\theta\, T_{11} - \sin\theta\, T_{12} \\ 0 = \cos\theta\, T_{10} - i\sin\theta\, T_{13} \\ 0 = -i\cos\theta\, T_{13} - \sin\theta\, T_{10} \\ 0 = i\cos\theta\, T_{12} + i\sin\theta\, T_{11} \quad . \end{cases}$$

From these equations

$$(7.9a)\qquad \begin{cases} T_{10} = T_{13} = 0 \quad , \quad T_{11} = b(\theta)\cos\theta \\ T_{12} = -\, b(\theta)\sin\theta \quad . \end{cases}$$

On using (7.5d) and Appendix I

$$(7.10)\qquad \begin{cases} b(\theta) = \sin\theta\, T_{21} + \cos\theta\, T_{22} \\ 0 = \sin\theta\, T_{20} + i\cos\theta\, T_{23} \\ 0 = -i\sin\theta\, T_{23} + \cos\theta\, T_{20} \\ 0 = i\sin\theta\, T_{22} - i\cos\theta\, T_{21} \quad , \end{cases}$$

from which

$$(7.10a)\qquad \begin{cases} T_{22} = b(\theta)\cos\theta \quad , \quad T_{21} = b(\theta)\sin\theta \\ T_{20} = T_{23} = 0 \quad . \end{cases}$$

On using (7.7), (7.8), (7.9a), and (7.10a) we see that we have

$$(7.11)\qquad T_{ij} = b(\theta)\, a^{i}{}_{j}$$

Now to find $b(\theta)$, we shall use our condition (b) of page 15. The condition of reality yields the fact that $b(\theta)$ is real. The group property of the Lorentz transformation in the plane leads to the results

$$(7.12) \qquad \begin{cases} b(\theta + \phi) = b(\theta)\, b(\phi) \quad , \quad b(0) = 1 \\ b(-\theta) = b(\theta)^{-1} \quad , \end{cases}$$

i.e., the function $b(\theta)$ is a one-dimensional representation of the plane rotation. The only <u>real</u> and <u>continuous</u> solution of (7.12) is

$$(7.13) \qquad b(\theta) = e^{\kappa\theta} \quad , \quad \kappa \text{ real} \quad .$$

We note from (7.13) and (7.11) that T will not be a single valued representation unless $\kappa = 0$ and $b(\theta) = 1$. On accepting this requirement, we have

$$(7.14) \qquad T_{ij} = a^i{}_j$$

for any plane rotation of the x^1, x^2 axis coordinates.

Having established (7.14) for such a rotation of the x^1, x^2 axis, we can now find the corresponding transformation S from (7.5a). We have

$$(7.15) \qquad S = T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If we now use Appendix I, (7.14), and (7.4), we can show that equations (7.5b), (7.5c) and (7.5d) also give the same result for S.

We further note that S as given by (7.15) is orthogonal.

Hence we have proved that plane rotations at the x^1, x^2 axis generate unique transformations S and T, such that S is orthogonal and T is real and is given by (7.14).

The procedure above can be used to prove that <u>all</u> plane rotations of two space axes generate an orthogonal transformation S and a real transformation T given by (7.14).

Let us now consider a plane rotation involving the time axis x^0 and a space axis, say x^1,

$$\text{(7.16)} \qquad \begin{cases} x^{0'} = x^0 \cosh \theta + x^1 \sinh \theta \\ x^{1'} = x^0 \sinh \theta + x^1 \cosh \theta \quad . \end{cases}$$

In this case we have

$$\text{(7.17)} \qquad \begin{cases} a^0{}_0 = \cosh \theta = a^1{}_1 \\ a^0{}_1 = \sinh \theta = a^1{}_0 \\ a^2{}_2 = a^3{}_3 = 1 \\ \text{all other } a^i{}_j = 0 \quad . \end{cases}$$

As usual, the primed system of coordinates may be thought to be moving with a velocity v with respect to the unprimed system where

$$\text{(7.18)} \qquad \sinh \theta = \frac{v}{\sqrt{1-v^2}} \quad , \qquad \text{(velocity of light = 1)} \; .$$

An analysis similar to that used in deriving (7.11) and (7.13) leads to

$$\text{(7.19)} \qquad T_{ij} = e^{\lambda\theta} \, a^i{}_j \quad ,$$

where λ is real. If we use requirement (c) of page 15, we see that λ must be zero, for if $\lambda \neq 0$, then a change in the sign of v changes the sign of θ (equation (7.18)) and consequently the

absolute values of the components of $\Phi' = T\,\Phi$. Hence (7.14) holds also for rotations in the x^0, x^1 plane. Each of the equations (7.1) yields the same result for S, namely,

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\theta & -i\sinh\theta \\ 0 & 0 & i\sinh\theta & \cosh\theta \end{pmatrix} \tag{7.20}$$

Again we see that S is orthogonal.

In a similar way all plane rotations involving the x^0 axis and a space axis yield a transformation (7.14) for T and an orthogonal matrix for S.

To summarize: Under the conditions imposed on T described above, <u>every rotation in a plane yields the result</u> (7.2) <u>for</u> T. <u>Furthermore</u>, S <u>as calculated by any of the equations</u> (7.1) <u>gives the same result, namely an orthogonal matrix associated with the rotation</u>.

Let us now consider Lorentz transformations which can be represented as a product of two plane rotations $L = \overset{2}{L}\,\overset{1}{L}$. The relation between the corresponding transformation matrices is

$$a^i{}_j = \overset{2}{a}{}^i{}_k\,\overset{1}{a}{}^k{}_j \quad . \tag{7.21}$$

On writing

$$T = T\,\overset{1}{T}{}^{-1}\,\overset{1}{T} = R\,\overset{1}{T} \quad , \tag{7.22}$$

$$R = T\,\overset{1}{T}{}^{-1} \quad , \tag{7.22a}$$

where $\overset{1}{T}_{ij} = \overset{1}{a}{}^i{}_j$ in accordance with our proof for rotations in a plane, equations (7.1) become

$$(7.23)\qquad S = \overset{2}{a}{}^{(i)}{}_{k}\,\overset{1}{a}{}^{k}{}_{j}\,\alpha^{(i)}\,R\,\overset{1}{T}\,\alpha^{j}$$

$$= \overset{2}{a}{}^{(i)}{}_{k}\,\alpha^{(i)}\,R\,\alpha^{k}\,\overset{1}{a}{}^{k}{}_{j}\,\alpha^{k}\,\overset{1}{T}\,\alpha^{j}$$

$$= \overset{2}{a}{}^{(i)}{}_{k}\,\alpha^{(i)}\,R\,\alpha^{k}\,\overset{1}{S}\ .$$

Let now,

$$(7.24)\qquad S\,\overset{1}{S}{}^{-1} = W\ .$$

then

$$(7.25)\qquad W = \overset{2}{a}{}^{(i)}{}_{k}\,\alpha^{(i)}\,R\,\alpha^{k}\ ,$$

where from the conditions on S and $\overset{1}{S}$, we have

$$(7.26)\qquad W_{01} = W_{02} = W_{03} = 0\ .$$

By the very same reasoning which enabled us to evaluate T for a plane rotation, we see that equation (7.25) with conditions (7.26) on W leads to the following result for R:

$$(7.27)\qquad \begin{cases} R = e^{k\theta}\,\overset{2}{T}\ , \\ W_{00} = e^{k\theta} \end{cases}$$

where $\overset{2}{T}$ is the operator T associated with the plane rotation whose transformation matrix is $\overset{2}{a}{}^{i}{}_{j}$ and θ is the angle of rotation. But from (7.22a)

$$(7.28)\qquad T = R\,\overset{1}{T} = e^{k\theta}\,\overset{2}{T}\,\overset{1}{T}{}^{2}\ .$$

From conditions (b) or (c) of page 15, we see that $k = 0$ and we have

$$(7.29)\qquad T = \overset{2}{T}\,\overset{1}{T}\ .$$

Since

(7.30) $$\overset{2}{T}_{ij} = \overset{2}{a}{}^{i}{}_{j} \quad , \quad \overset{1}{T}_{ij} = \overset{1}{a}{}^{i}{}_{j} \quad ,$$

it is clear from (7.21) that

(7.31) $$T_{ij} = \overset{2}{T}_{ik}\overset{1}{T}_{kj} = a^{i}{}_{j} \quad .$$

Hence we have proved (7.2) for Lorentz transformations which are products of two plane transformations.

On using (7.27) with $\theta = 0$ in (7.25) it is seen that

(7.32) $$W = \overset{2}{S}$$

where $\overset{2}{S}$ is the matrix S associated with the plane relation given by $\overset{2}{a}{}^{i}{}_{j}$. Finally, equation (7.24) yields the result

(7.33) $$S = \overset{2}{S}\overset{1}{S}$$

and is given by all four equations (7.1). Since $\overset{2}{S}$, $\overset{1}{S}$ are orthogonal, so is S.

This mode of reasoning can be extended to Lorentz transformations which are a product of n plane rotations. In this general case we obtain

(7.34) $$T_{ij} = a^{i}{}_{j} \quad ,$$

(7.35) $$T = \overset{n}{T}\;\overset{n-1}{T} \quad \cdots \quad \overset{2}{T}\overset{1}{T}$$

(7.36) $$S = \overset{n}{S}\;\overset{n-1}{S} \quad \cdots \quad \overset{2}{S}\overset{1}{S} \quad ,$$

where $\overset{i}{T}$ and $\overset{i}{S}$ are the matrices associated with the i-th plane rotation. Further, in the process of proving (7.36), one shows

that all four equations of (7.1) give the same S, which, in addition, is an orthogonal matrix.

Inasmuch as all proper Lorentz transformations can be written as a product of n plane transformations ($n \leq 6$), we have accomplished our purpose of showing all the equations (7.1) yield a single solution for an orthogonal matrix S and a real matrix T where $T_{ij} = a^i{}_j$.

8. Spin of the Electromagnetic Field.

It has been shown that when there are no external sources, Maxwell's equations in spinor form are a special case of Dirac's equations for mass zero. The transformation properties of Maxwell's and Dirac's equations differ, however. Whereas the transformation properties of Dirac's equations lead to the result that the Dirac field has a spin $\frac{1}{2}$, the transformation properties of the Maxwell field will be shown to yield a spin 1.

As usual [4], the operator associated with component of angular momentum around any axis $\underset{\sim}{x}$ is defined in terms of infinitesimal rotations [5] in the plane perpendicular to $\underset{\sim}{x}$ about $\underset{\sim}{x}$. Let $\delta\psi$ be the change in ψ due to an infinitesimal rotation $\delta\theta$ about the axis $\underset{\sim}{x}$. Then the angular momentum operator component about $\underset{\sim}{x}$, denoted by $M_{\underset{\sim}{x}}$, is defined by

$$\text{(8.1)} \qquad \frac{1}{i}\,\delta\psi = M_{\underset{\sim}{x}}\,\psi\,\delta\theta \quad .$$

In general one can write

$$\text{(8.2)} \qquad M_{\underset{\sim}{x}} = \sigma_{\underset{\sim}{x}} + \frac{1}{i}\frac{\partial}{\partial\theta} \quad ,$$

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[4] P.A.M. Dirac, The Principles of Quantum Mechanics, Oxford, Third Edition (1947), ¶ 35.

[5] In the definition of angular momentum only space-like rotations are considered.

where $\underset{\sim}{\sigma_x}$ is a matrix acting on the components of ψ and the differential operator $\frac{\partial}{\partial\theta}$ acts on the arguments of each of the components of ψ, i.e.

$$(8.3) \qquad \frac{1}{i}\frac{\partial}{\partial\theta}\psi = \frac{1}{i}\begin{pmatrix} \frac{\partial\psi_0}{\partial\theta} \\ \frac{\partial\psi_1}{\partial\theta} \\ \frac{\partial\psi_2}{\partial\theta} \\ \frac{\partial\psi_3}{\partial\theta} \end{pmatrix}$$

The operator $\sigma_{\underset{\sim}{x}}$ is called the "spin operator component" about the axis $\underset{\sim}{x}$.

In our particular case we know that for any space-like rotation we have

$$(8.4) \qquad S = T \quad .$$

To find σ_1, σ_2, σ_3 which are the spin operator components about the x^1, x^2, x^3 axes, we consider small rotations about these axes and calculate $\delta\psi = (S - I)\psi$ to first order in the rotation. On using (8.1) and (8.2) we can obtain our results, which are

$$(8.5a) \qquad \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$(8.5b) \qquad \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$(8.5c) \qquad \sigma_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It is easily seen that σ_1, σ_2, σ_3 are all Hermitian operators whose eigenvalues are +1, -1, 0, 0. If one restricts oneself to working upon column vectors such that $\psi_0 = 0$, as we always do, then only one zero appears as an eigenvalue of σ_i, $(i = 1, 2, 3)$.

We also note

$$(8.6) \qquad (\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2 = 2 = \ell(\ell + 1) \ , \quad (\ell = 1) \ .$$

Equation (8.6) together with the eigenvalues of σ_i indicate that the electromagnetic field is indeed a field of spin 1.

It can also be shown that

$$(8.7) \qquad \begin{cases} \sigma_2 \sigma_3 - \sigma_3 \sigma_2 = i\sigma_1 \\ \sigma_3 \sigma_1 - \sigma_1 \sigma_3 = i\sigma_2 \\ \sigma_1 \sigma_2 - \sigma_2 \sigma_1 = i\sigma_3 \end{cases}$$

Equations (8.7) are sometimes written

$$(8.8) \qquad \underset{\sim}{\sigma} \times \underset{\sim}{\sigma} = i\underset{\sim}{\sigma} \ ,$$

because the left hand sides of (8.7) resemble the components of a cross product in form. Equations (8.7) and (8.8) are familiar ones in the theory of spin.

One can also show that any component of the total angular momentum commutes with the Hamiltonian

(8.9) $$H = \frac{1}{i}\left[\alpha^1 \frac{\partial}{\partial x^1} + \alpha^2 \frac{\partial}{\partial x^2} + \alpha^3 \frac{\partial}{\partial x^3}\right]$$

and that the square of the angular momentum also commutes with this quantity. We refrain from going into details.

On comparing expressions (8.5) for σ_i with (3.4) for α^i we see that

(8.10) $$\sigma_i = \rho \alpha^i \rho \quad , \qquad (i = 1, 2, 3) \quad ,$$

where ρ is the matrix

(8.11) $$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is clear that ρ is a projection operator, which when acting on a four component column vector replaces the top component by zero. The vectors ψ which we use in the description of the Maxwell field are eigenstates of

(8.12) $$\rho\psi = \psi \quad ,$$

because $\psi_0 = 0$. Therefore, if we restrict ourselves to ψ vectors of the Maxwell field, we have on using (8.12)

(8.13) $$\sigma_i \psi = \rho \alpha^i \rho \psi = \rho \alpha^i \psi \quad ,$$

or

(8.14) $$\sigma_i = \rho \alpha^i$$

when acting on a Maxwell field.

It should also be noted that our σ_i are essentially the same as ζ^i of Molière (Section 2).

On using (8.10) and (8.14) one can show that the operator $\underset{\sim}{\sigma}\cdot\underset{\sim}{P}$, defined by

$$(8.15) \qquad \underset{\sim}{\sigma}\cdot\underset{\sim}{P} = \frac{1}{i}\left[\sigma_1 \frac{\partial}{\partial x^1} + \sigma_2 \frac{\partial}{\partial x^2} + \sigma_3 \frac{\partial}{\partial x^3}\right] \quad ,$$

commutes with H. This result is analogous to a situation which occurs with Dirac's equations for the electron.

9. Improper Lorentz Transformations.

Thus far we have restricted our discussion to proper Lorentz transformations which are defined to be transformations whose transformations matrices satisfy

$$(9.1) \qquad \begin{cases} \det (a^i{}_j) = 1 \\ a^0{}_0 > 0 \quad . \end{cases}$$

Such transformations have the property that they can be obtained by a continuous process from the identity.

Improper Lorentz transformations may be regarded as products of proper Lorentz transformations and certain simple improper transformations. We shall discuss how ψ and $\underline{\Phi}$ transform under these simple improper transformations.

I. Space and Time Reversal.

Maxwell's equations in spinor notation are

$$(9.1) \qquad -\frac{1}{i}\,\alpha^j \frac{\partial}{\partial x^j}\,\psi = -4\pi\underline{\Phi} \quad .$$

When the space and time coordinates are all reversed, we have

(9.2) $$x^{i'} = - x^{i} \quad , \qquad (i = 0, 1, 2, 3) \quad .$$

Equation (9.1) becomes

(9.3) $$\frac{1}{i} \alpha^{j} \frac{\partial}{\partial x^{j'}} \psi = -4\pi \underline{\Phi}$$

We wish to introduce vectors ψ' and $\underline{\Phi}'$ such that under a transformation (9.2) we have instead of (9.3)

(9.4) $$- \frac{1}{i} \alpha^{j} \frac{\partial}{\partial x^{j'}} \psi' = -4\pi \underline{\Phi}'$$

We see that (9.3) becomes (9.4) under the transformation

(9.5) $$\psi' = b\psi \quad , \quad \underline{\Phi}' = -b\underline{\Phi} \quad ,$$

where b is any real constant. Since we want two reversals to be the identity transformation, we have $b^2 = 1$ or

(9.5a) $$b = \pm 1 \quad .$$

In particular, if we have any a priori reason to require the components of $\underline{\Phi}$ to transform like the 4-vector x^i we should take $b = 1$.

It is to be noted that $\underline{\Phi}$ and ψ still transform linearly. We could have written

(9.6) $$\begin{cases} \psi' = S\psi \\ \underline{\Phi}' = T\underline{\Phi} \end{cases}$$

and required S and T to satisfy

(9.7) $$\begin{cases} S = -\alpha^{(i)} \; T \; \alpha^{(i)} \quad , \quad (i = 0, 1, 2, 3) \\ S_{01} = S_{02} = S_{03} = 0 \\ T \text{ a real matrix} \end{cases}$$

On solving for S and T, as for proper transformations, we would find

$$(9.8)\qquad \begin{cases} T = bI \\ S = -bI \end{cases},$$

as before. Hence, the case of time ans space reversal can be handled in essentially the same way as for proper transformations.

II. Time Reversal.

Let

$$(9.9)\qquad \begin{cases} x^{0'} = -x^{0}, \\ x^{i'} = x^{i}, \qquad (i = 1, 2, 3) \end{cases}$$

Under the transformation (9.9), equation (9.1) becomes

$$(9.10)\qquad \left[\frac{1}{i}\frac{\partial}{\partial x^{0'}} - \frac{1}{i}\alpha^{1}\frac{\partial}{\partial x^{1'}} - \frac{1}{i}\alpha^{2}\frac{\partial}{\partial x^{2'}} - \frac{1}{i}\alpha^{3}\frac{\partial}{\partial x^{3'}}\right]\psi = -4\pi\Phi$$

If we try to obtain (9.4) from (9.10) using linear transformations of the form (9.6), we find that the four equations for S and T analogous to (9.7) are inconsistent. Hence this procedure cannot be used.

We must therefore think in terms of more general transformations.

We require of this more general transformations not only that (9.4), but also the complex conjugate of (9.4), namely

$$(9.4a)\qquad \frac{1}{i}\alpha^{j*}\frac{\partial}{\partial x^{j'}}\psi'^{*} = -4\pi\Phi ,$$

be obtained from (9.10) and its complex conjugate

$$(9.10a)\quad \left[-\frac{1}{i}\frac{\partial}{\partial x^{0'}} + \frac{1}{i}\alpha^{1*}\frac{\partial}{\partial x^{1'}} + \frac{1}{i}\alpha^{2*}\frac{\partial}{\partial x^{2'}} + \frac{1}{i}\alpha^{3*}\frac{\partial}{\partial x^{3'}}\right]\psi^* = -4\pi\underline{\Phi}$$

where α^{i*} is the matrix which is the complex conjugate of α^i. We, of course, also require $\underline{\Phi}'$ have only real components, and $\psi_0' = \psi_0'^* = 0$.

A transformation that may conceivably turn (9.10) and (9.10a) into (9.4) and (9.4a) is

$$(9.11)\qquad \begin{aligned} \psi' &= S\psi^* \\ \underline{\Phi}' &= T\underline{\Phi} \end{aligned}$$

subject to the conditions

$$(9.11a)\qquad \begin{cases} S_{01} = S_{02} = S_{03} = 0 \\ T \text{ a real transformation} \end{cases}$$

Transformations for ψ of the type (9.11) are called anti-linear. The equations for S and T become, after substituting (9.11) into (9.10a)

$$(9.12)\qquad \begin{cases} S = T \\ S = -\alpha^{(i)}\, S\, \alpha^{(i)*} \quad , \quad (i = 1, 2, 3) \end{cases}$$

These equations can indeed be solved under conditions (9.11a) and one obtains, on using the condition that two time reversals should be the identity transformation

$$(9.13)\qquad S = T = b \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(9.13a) $$b = +1 \quad \text{or} \quad -1 \quad .$$

If we take b = 1, so that Φ transforms like a true 4-vector, we have

(9.14)
$$\begin{aligned} \psi_1' &= \psi_1^* \\ \psi_2' &= \psi_2^* \\ \psi_3' &= \psi_3^* \end{aligned}$$

or taking real and imaginary parts

(9.14a)
$$\begin{cases} E_x' = -E_x & H_x' = H_x \\ E_y' = -E_y & H_y' = H_y \\ E_z' = -E_z & H_z' = H_z \end{cases}$$

i.e. the electric field reverses sign under a time reversal transformation but the magnetic field is unchanged.

III. Space Inversion.

In this case

(9.15)
$$\begin{aligned} x^{0'} &= x^0 \\ x^{i'} &= -x^i \quad , \qquad (i = 1, 2, 3) \quad . \end{aligned}$$

Under this transformations of coordinates, Maxwell's equations become

(9.16)
$$\left[-\frac{1}{i}\frac{\partial}{\partial x^{0'}} + \frac{1}{i}\alpha^1\frac{\partial}{\partial x^{1'}} + \frac{1}{i}\alpha^2\frac{\partial}{\partial x^{2'}} + \frac{1}{i}\alpha^3\frac{\partial}{\partial x^{3'}}\right]\psi = -4\pi\Phi$$
$$\left[\frac{1}{i}\frac{\partial}{\partial x^{0'}} - \frac{1}{i}\alpha^1\frac{\partial}{\partial x^{1'}} - \frac{1}{i}\alpha^{2*}\frac{\partial}{\partial x^{2'}} - \frac{1}{i}\alpha^{3*}\frac{\partial}{\partial x^{3'}}\right]\psi^* = -4\pi\Phi \, .$$

In order to get (9.16) into the form (9.4) and (9.4a), we again assume an antilinear transformation of the form (9.11) under

conditions (9.11a). The equations for S and T are

$$(9.17) \qquad \begin{cases} S = -T \\ S = -\alpha^{(i)}\, S\, \alpha^{(i)*} , \qquad (i = 1, 2, 3) \end{cases}$$

The solution of (9.17) is

$$(9.18) \qquad S = -T = b \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(9.18a) \qquad b = +1 \quad \text{or} \quad -1 \quad .$$

If we pick b = 1, which corresponds, as in the case of time reversal, to $\bar{\Phi}$ transforming like a four vector we see that ψ transforms as in (9.14), or $\underset{\sim}{E}$ and $\underset{\sim}{H}$ as in (9.14a). Under a space inversion $\underset{\sim}{E}$ behaves like a vector and $\underset{\sim}{H}$ like a pseudo-vector.

IV. Space Reflections.

In this case we invert a single space axis, the x^1-axis for example. We have

$$(9.19) \qquad \begin{cases} x^{0\prime} = x^0 \\ x^{2\prime} = x^2 \\ x^{3\prime} = x^3 \\ x^{1\prime} = -\, x^1 \end{cases}$$

On assuming a transformation of the form (9.11) with conditions (9.11a) we have as the equations for S and T

$$(9.20)\qquad \begin{cases} S = -T \\ S = \alpha^2\, S\, \alpha^{2*} \\ S = \alpha^3\, S\, \alpha^{3*} \\ S = -\,\alpha^1\, S\, \alpha^{1*} \end{cases}$$

The solutions of S and T are

$$(9.21)\qquad S = -T = b \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$(9.21a)\qquad b = +1 \quad \text{or} \quad -1 \quad .$$

If $b = +1$, and Φ therefore transforms as a 4-vector, we have

$$(9.22)\qquad \begin{cases} \psi_1' = \psi_1^* \\ \psi_2' = -\,\psi_2^* \\ \psi_3' = -\,\psi_3^* \end{cases}$$

or

$$(9.22a)\qquad \begin{cases} E_x' = -\,E_x & \qquad H_x' = H_x \\ E_y' = E_y & \qquad H_y' = -\,H_y \\ E_z' = E_z & \qquad H_z' = -\,H_z \end{cases}$$

Thus $\underset{\sim}{E}$ transforms like a vector and $\underset{\sim}{H}$ like a pseudo-vector.

10. <u>The Equation of Continuity, the Vector Potential, and the Lorentz Condition</u>.

We can use the fact that the operator $-\frac{1}{i}\alpha^j \frac{\partial}{\partial x^j}$ is a square root of the wave operator and that $\psi_0 = 0$ to obtain several well-known results in a simple way. From the commutation relations (3.5a) for the α^i operators, which we repeat below,

$$(10.1) \qquad \begin{cases} \alpha^i \alpha^j + \alpha^j \alpha^i = 2\delta_{ij} I \ , & (i = 1, 2, 3) \\ \alpha^0 = I \ . & \end{cases}$$

we have the following identity which is a familiar one in the Dirac theory of the electron:

$$(10.2) \qquad \left[-\frac{1}{i}\alpha^j \frac{\partial}{\partial x^j}\right]\left[-\frac{1}{i}\alpha^k \frac{\partial}{\partial x_k}\right] = \left[-\frac{1}{i}\alpha^k \frac{\partial}{\partial x_k}\right]\left[-\frac{1}{i}\alpha^j \frac{\partial}{\partial x^j}\right]$$

$$= -\frac{\partial}{\partial x^j}\frac{\partial}{\partial x_j}$$

$$= \nabla^2 - \frac{\partial^2}{\partial t^2} \ .$$

Hence, on applying the operator $-\frac{1}{i}\alpha^k \frac{\partial}{\partial x_k}$ to the left in Maxwell's equations

$$(10.3) \qquad -\frac{1}{i}\alpha^j \frac{\partial}{\partial x^j}\psi = -4\pi\underline{\Phi} \ ,$$

we obtain

$$(10.4) \qquad \frac{\partial}{\partial x^j}\frac{\partial}{\partial x_j}\psi = -\frac{4\pi}{i}\alpha^j \frac{\partial}{\partial x_j}\underline{\Phi} \ .$$

If $\underline{\Phi} = 0$, i.e. there are no external sources, then it is clear from (10.2) each component of ψ obeys the wave equation. Hence each component of the electric and magnetic fields obeys the wave equation.

If $\underline{\Phi}$ is not zero, we use the fact that $\psi_0 = 0$ to set the top component of the right hand side of (10.4) equal to zero. On using the representation (3.4) for the α's we immediately obtain the equation of continuity

(10.5) $$\frac{\partial \Phi_0}{\partial x^0} + \frac{\partial \Phi_1}{\partial x^1} + \frac{\partial \Phi_2}{\partial x^2} + \frac{\partial \Phi_3}{\partial x^3} = \frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = 0 \quad .$$

The vector potential A, which we take to be a column vector

(10.6) $$A = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

is defined in the spinor notation by

(10.7) $$\psi = -\frac{1}{i}\alpha^k \frac{\partial}{\partial x_k} A$$

On substituting (10.7) into Maxwell's equations (10.3) and on using (10.2), we have

(10.8) $$-\frac{\partial}{\partial x^j}\frac{\partial}{\partial x_j} A = -4\pi\Phi$$

or in terms of components

(10.8a) $$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right) A_i = -4\pi\Phi_i$$

which are the usual equations for the vector potential. In (10.7) the requirement that $\psi_0 = 0$ leads to the Lorentz condition, for on setting the topmost component on the right with (10.7) equal to zero we have

(10.9) $$\frac{\partial}{\partial x^0} A_0 + \frac{\partial}{\partial x^1} A_1 + \frac{\partial}{\partial x^2} A_2 + \frac{\partial}{\partial x^3} A_3 = \frac{\partial}{\partial x^i} A_i = 0 \quad .$$

11. Acknowledgements.

The writer wishes to express his thanks to Professor K. O. Friedrichs and Dr. R. Kraichnan of New York University and Professor N. Marcuvitz of Brooklyn Polytechnic Institute for helpful discussions.

Appendix I

Matrix Elements $\alpha^i\ T\ \alpha^j$.

$$\text{(I-1)} \qquad \alpha^0 T \alpha^0 = T = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}$$

$$\text{(I-2)} \qquad \alpha^0 T \alpha^1 = T\alpha^1 = \begin{pmatrix} -T_{01} & -T_{00} & iT_{03} & -iT_{02} \\ -T_{11} & -T_{10} & iT_{13} & -iT_{12} \\ -T_{21} & -T_{20} & iT_{23} & -iT_{22} \\ -T_{31} & -T_{30} & iT_{33} & -iT_{32} \end{pmatrix}$$

$$\text{(I-3)} \qquad \alpha^0 T \alpha^2 = T\alpha^2 = \begin{pmatrix} -T_{02} & -iT_{03} & -T_{00} & iT_{01} \\ -T_{12} & -iT_{13} & -T_{10} & iT_{11} \\ -T_{22} & -iT_{23} & -T_{20} & iT_{21} \\ -T_{32} & -iT_{33} & -T_{30} & iT_{31} \end{pmatrix}$$

$$\text{(I-4)} \qquad \alpha^0 T \alpha^3 = T\alpha^3 = \begin{pmatrix} -T_{03} & iT_{02} & -iT_{01} & -T_{00} \\ -T_{13} & iT_{12} & -iT_{11} & -T_{10} \\ -T_{23} & iT_{22} & -iT_{21} & -T_{20} \\ -T_{33} & iT_{32} & -iT_{31} & -T_{30} \end{pmatrix}$$

$$(\text{I-5}) \qquad \alpha^1 T \alpha^0 = \alpha^1 T = \begin{pmatrix} -T_{10} & -T_{11} & -T_{12} & -T_{13} \\ -T_{00} & -T_{01} & -T_{02} & -T_{03} \\ -iT_{30} & -iT_{31} & -iT_{32} & -iT_{33} \\ iT_{20} & iT_{21} & iT_{22} & iT_{23} \end{pmatrix} .$$

$$(\text{I-6}) \qquad \alpha^1 T \alpha^1 = \begin{pmatrix} T_{11} & T_{10} & -iT_{13} & iT_{12} \\ T_{01} & T_{00} & -iT_{03} & iT_{02} \\ iT_{31} & iT_{30} & T_{33} & -T_{32} \\ -iT_{21} & -iT_{20} & -T_{23} & T_{22} \end{pmatrix} .$$

$$(\text{I-7}) \qquad \alpha^1 T \alpha^2 = \begin{pmatrix} T_{12} & iT_{13} & T_{10} & -iT_{11} \\ T_{02} & iT_{03} & T_{00} & -iT_{01} \\ iT_{32} & -T_{33} & iT_{30} & T_{31} \\ -iT_{22} & T_{23} & -iT_{20} & -T_{21} \end{pmatrix} .$$

$$(\text{I-8}) \qquad \alpha^1 T \alpha^3 = \begin{pmatrix} T_{13} & -iT_{12} & iT_{11} & T_{10} \\ T_{03} & -iT_{02} & iT_{01} & T_{00} \\ iT_{33} & T_{32} & -T_{31} & iT_{30} \\ -iT_{23} & -T_{22} & T_{21} & -T_{20} \end{pmatrix} .$$

$$(\text{I-9}) \qquad \alpha^2 T \alpha^0 = \alpha^2 T = \begin{pmatrix} -T_{20} & -T_{21} & -T_{22} & -T_{23} \\ iT_{30} & iT_{31} & iT_{32} & iT_{33} \\ -T_{00} & -T_{01} & -T_{02} & -T_{03} \\ -iT_{10} & -iT_{11} & -iT_{12} & -iT_{13} \end{pmatrix}.$$

$$(\text{I-10}) \qquad \alpha^2 T \alpha^1 = \begin{pmatrix} T_{21} & T_{20} & -iT_{23} & iT_{22} \\ -iT_{31} & -iT_{30} & -T_{33} & T_{32} \\ T_{01} & T_{00} & -iT_{03} & iT_{02} \\ iT_{11} & iT_{10} & T_{13} & -T_{12} \end{pmatrix}.$$

$$(\text{I-11}) \qquad \alpha^2 T \alpha^2 = \begin{pmatrix} T_{22} & iT_{23} & iT_{20} & -iT_{21} \\ -iT_{32} & T_{33} & -iT_{30} & -T_{31} \\ T_{02} & iT_{03} & T_{00} & -iT_{01} \\ iT_{12} & -T_{13} & iT_{10} & T_{11} \end{pmatrix}.$$

$$(\text{I-12}) \qquad \alpha^2 T \alpha^3 = \begin{pmatrix} T_{23} & -iT_{22} & iT_{21} & T_{20} \\ -iT_{33} & -T_{32} & T_{31} & -iT_{30} \\ T_{03} & -iT_{02} & iT_{01} & T_{00} \\ iT_{13} & T_{12} & -T_{11} & iT_{10} \end{pmatrix}.$$

(I-13) $$\alpha^3 T \alpha^0 = \alpha^3 T = \begin{pmatrix} -T_{30} & -T_{31} & -T_{32} & -T_{33} \\ -iT_{20} & -iT_{21} & -iT_{22} & -iT_{23} \\ iT_{10} & iT_{11} & iT_{12} & iT_{13} \\ -T_{00} & -T_{01} & -T_{02} & T_{03} \end{pmatrix} .$$

(I-14) $$\alpha^3 T \alpha^1 = \begin{pmatrix} T_{31} & T_{30} & -iT_{33} & iT_{32} \\ iT_{21} & iT_{20} & T_{23} & -T_{22} \\ -iT_{11} & -iT_{10} & -T_{13} & T_{02} \\ T_{01} & T_{00} & -iT_{03} & iT_{02} \end{pmatrix} .$$

(I-15) $$\alpha^3 T \alpha^2 = \begin{pmatrix} T_{32} & iT_{33} & T_{30} & -iT_{31} \\ iT_{22} & -T_{23} & iT_{20} & T_{21} \\ -iT_{12} & T_{13} & -iT_{10} & -T_{11} \\ T_{02} & iT_{03} & T_{00} & -iT_{01} \end{pmatrix} .$$

(I-16) $$\alpha^3 T \alpha^3 = \begin{pmatrix} T_{33} & -iT_{32} & iT_{31} & T_{30} \\ iT_{23} & T_{22} & -T_{21} & iT_{20} \\ -iT_{13} & -T_{12} & T_{11} & -iT_{10} \\ T_{03} & iT_{02} & iT_{01} & T_{00} \end{pmatrix} .$$

Appendix II

Elements of Matrix S for a Proper Lorentz Transformation.

(II-1) $S_{00} = 1$

$S_{01} = S_{02} = S_{03} = S_{10} = S_{20} = S_{30} = 0$.

(II-2) $S_{11} = (a^1{}_1\, a^0{}_0 - a^1{}_0\, a^0{}_1) + i(a^1{}_2\, a^0{}_3 - a^1{}_3\, a^0{}_2)$.

(II-3) $S_{12} = (a^1{}_2\, a^0{}_0 - a^1{}_0\, a^0{}_2) + i(a^1{}_3\, a^0{}_1 - a^1{}_1\, a^0{}_3)$.

(II-4) $S_{13} = (a^1{}_3\, a^0{}_0 - a^1{}_0\, a^0{}_3) + i(a^1{}_1\, a^0{}_2 - a^1{}_2\, a^0{}_1)$.

(II-5) $S_{21} = (a^2{}_1\, a^0{}_0 - a^2{}_0\, a^0{}_1) + i(a^2{}_2\, a^0{}_3 - a^2{}_3\, a^0{}_2)$.

(II-6) $S_{22} = (a^2{}_2\, a^0{}_0 - a^2{}_0\, a^0{}_2) + i(a^2{}_3\, a^0{}_1 - a^2{}_1\, a^0{}_3)$.

(II-7) $S_{23} = (a^2{}_3\, a^0{}_0 - a^2{}_0\, a^0{}_3) + i(a^2{}_1\, a^0{}_2 - a^2{}_2\, a^0{}_1)$.

(II-8) $S_{31} = (a^3{}_1\, a^0{}_0 - a^3{}_0\, a^0{}_1) + i(a^3{}_2\, a^0{}_3 - a^3{}_3\, a^0{}_2)$.

(II-9) $S_{32} = (a^3{}_2\, a^0{}_0 - a^3{}_0\, a^0{}_2) + i(a^3{}_3\, a^0{}_1 - a^3{}_1\, a^0{}_3)$.

(II-10) $S_{33} = (a^3{}_3\, a^0{}_0 - a^3{}_0\, a^0{}_3) + i(a^3{}_1\, a^0{}_2 - a^3{}_2\, a^0{}_1)$.

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